Auctions and Mechanism Design

Maryam Kamgarpour

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1 Outline

- First-price and second-price auctions
- Dominant strategy incentive compatible Nash equilibrium
- VCG Mechanism
- Collusion and shill bidding

2 Introduction

This is a peak into auction theory and mechanism design. Auctions and mechanisms are used in many real-world scenarios including electricity markets, spectrum auctions (for governments selling frequency bands to different communication companies), kidney exchange auctions, transportation networks, online ad markets, e-Bay, facebook, ... We discuss the first-price (also known as pay-as-bid) and second-price auctions using game theory. We then consider auctions of multiple possibly divisible items. We discuss desired properties of a mechanism and discuss a generalization of the second-price auction known as the Vickrey-Clarke-Groves (VCG) mechanism.

Exercise 1. Form groups of two. Discuss where you have come across auctions. Have you participated in any auctions? Do you think an auction is a game?

2.1 First-price and second-price sealed-bid auctions

Consider a reverse auction motivated by those arising in electricity markets. Note that in a reverse auction, the auctioneer wants to purchase an item. Hence, it picks the bid with the lowest bid price (in a forward auction on the other hand, the auctioneer is selling an item and hence, she will give the item to the player with the highest bid.) In a first-price reverse auction, the player with the lowest bid wins and gets paid the price she bids. In a second-price auction, the bidder with the lowest bid wins (as before) but gets paid the price of the second lowest bid.

Exercise 2. Form groups of four or five. One of you is a power system operator, that is, the auctioneer. The rest are electricity providers. The operator wants to purchase 1 unit of energy. As the operator, you decide which auction you will run and you will announce it to the participants. As a provider, you each decide how much you will offer for the 1 unit of energy (do not discuss with the other providers). Suppose that you can be a *coal, gas, hydro or wind* energy provider. Your production cost depends on which type of provider you are and your cost can be anything from 0 to 100. Write your bid price on a piece of paper and pass it on to the auctioneer privately. The auctioneer decides the winner and announces the payment made to the winner (does not announce the winner).

As you notice in the exercise, from the players' perspective it is difficult to determine a bidding strategy to maximize profit. From the auctioneer's perspective, it is difficult to know how to set the auction rule to maximize the auction's revenue. The latter is an instance of *mechanism design*.

Can we use noncooperative game theory we studied so far to analyze both problems? Yes. indeed each player is a profit maximizing agent who does not cooperate with others as it can have no means of communication - not even aware of presence of other players (note that sometimes players try to signal each other even without knowing others in the game. This can result in collusion and shill bidding and for now, we ignore these issues). In particular, if we could analyze Nash equilibria of various mechanisms, then we could figure out optimal bidding strategies, profits of each participant and the auctioneer's total revenue. Furthermore, the auctioneer could understand its total revenue under different mechanisms.

2.2 Equilibria in auctions

Let us fix the number of auction participants (players) to N. Player j's true (private) value of the item is denoted by $t^j \in \mathbb{R}_+$ and her bid is denoted by $x^j \in \mathbb{R}_+$. The vector $x = [x^1, x^2, \dots, x^N] \in \mathbb{R}_+^N$ denotes the bids of all players and $x^{-j} \in \mathbb{R}_+^{N-1}$ denotes the vector x with the player j's bid removed. The set of all players is denoted by $\mathcal{N}$. The payment made by each player to the auctioneer (or by the auctioneer to each player in case of a reverse auction) is denoted by $p^j \in \mathbb{R}$. The utility (also referred to as payoff or profit) of the winner(s) is given by $J^j = t^j - p^j$ in a forward auction and $J^j = p^j - t^j$ in a reverse auction. For non-winners, $J^j = 0$. If participant j bids $x^j = t^j$ we say she is bidding truthfully. If however, $x^j < t^j$ she is underbidding, whereas if $x^j > t^j$ she is overbidding.

Exercise 3. Formulate the first-price and second-price reverse auctions discussed in the above exercise each as a game. What are the strategy sets? What are the payoffs?

Solution. Simply define the winner selection and the payment rules as a function of the bids. *First-price auction*: the winner $j^\star \in \mathcal{N}$ is found as $j^\star = \arg\min_{j \in \mathcal{N}} x^j$. The winner is paid its bid price $p^{j^\star} = x^{j^\star}$, while other participants are paid zero. So, the utility of the winner $j^\star$ is $J^{j^\star} = t^{j^\star} - x^{j^\star}$, while other players have zero utility.

Second-price auction: the winner $j^\star$ is found as before, that is, $j^\star = \arg\min_{j \in \mathcal{N}} x^j$. The winner in this case is paid the second highest price: $p^{j^\star} = \min_{j \neq j^\star} x^j$, while other players are paid zero. So, the utility of the winner is $J^{j^\star} = t^{j^\star} - \min_{j \neq j^\star} x^j$.

The definition of Nash equilibrium in auctions is as it'd be in any other game - a given player has no incentive for unilateral deviation from her strategy, which in this case is her bid:

Definition 1. A bidding profile x is a Nash equilibrium if $\forall j$

$$J^j(x^j, x^{-j}) \geq J^j(\tilde{x}^j, x^{-j}), \quad \forall \tilde{x}^j \in \mathbb{R}_+.$$

A stronger notion of Nash equilibrium is a dominant strategy Nash equilibrium. It says that no matter what the other players choose, it is best to play x^j.

Definition 2. A bidding strategy is a dominant strategy Nash equilibrium if $\forall j$

$$J^j(x^j, x^{-j}) \geq J^j(\tilde{x}^j, x^{-j}), \quad \forall \tilde{x}^j,\ \forall x^{-j}.$$

Clearly, for a participant to make profit in a first-price reverse auction she must overbid (in forward auction she must underbid). This generally results in all participants overbidding, which is not good for the auctioneer. We are interested in mechanisms which incentivize participants to bid truthfully. It turns out that such mechanisms can be designed.

Proposition 1. *Truthful bidding is a dominant strategy Nash equilibrium in a second-price auction.*

Solution. Note that the proof is written for the forward (not reverse) auction but exact same argument holds for a reverse auction, with inequalities reversed.

Proof. Let us consider an arbitrary player $j \in \mathcal{N}$. Consider the following two cases: Case 1: overbidding; Case 2: underbidding. We show that both overbidding and underbidding strategies are weakly dominated by truthful bidding.

- Case 1 - overbidding: $t^j < x^j$. We have the following three possibilities:

  1. $\max_{i \neq j} x^i < t^j$: overbidding and truthful bidding both result in winning the auction with the same payoff.
  2. $\max_{i \neq j} x^i > x^j$: overbidding and truthful bidding both result in losing the auction with zero payoff.
  3. $t^j < \max_{i \neq j} x^i < x^j$: overbidding results in winning the auction but a negative payoff $J^j = t^j - \max_{i \neq j} x^i < 0$; truthful bidding results in losing the auction and having zero profit.

  Hence, in all the above cases, overbidding is not better than truthful bidding.

- Case 2 - underbidding, $x^j < t^j$. We have the following three possibilities:

  1. $\max_{i \neq j} x^i < x^j$: underbidding and truthful bidding both result in winning the auction with the same payoff.
  2. $\max_{i \neq j} x^i > t^j$: underbidding and truthful bidding both result in losing the auction with zero payoff.
  3. $x^j < \max_{i \neq j} x^i < t^j$: underbidding results in losing the auction; truthful bidding results in winning the auction with $J^j = t^j - \max_{i \neq j} x^i > 0$ utility.

We just showed that the utility of player j does not improve by deviating from truthful bidding. Since j and x^{-j} were arbitrary, this implies that truthful bidding is a dominant strategy Nash equilibrium. □

Exercise 4. Are there other Nash equilibria in a second-price auction? Is truthful bidding a strictly dominant strategy Nash equilibrium?

As we can see, the second-price auction has a beautiful property that each player's profit is maximized by bidding truthfully. This property of the mechanism is referred to as *incentive compatibility*. Intuitively, the above nice property of the second price auction holds because $\forall j \in \mathcal{N}$, the payment to the winner, namely $\max_{i \neq j} x^i$, does not depend on the winner's bid x^j. It seems it could be a good idea to use this mechanism. Let us consider a slightly generalized example for a reverse auction motivated by real-world electricity markets and see how the winner and the payment rule is determined in a second-price auction with multiple items for sale/buy.

Exercise 5. A user wants to purchase 800 KW of electrical power. There are three providers. Player 1 offers 400 KW at \$25. Player 2 offers 600 KW at \$30 and 400 KW at \$24 conditional bids, meaning that only one of the two bids can be accepted. Player 3 offers 400 KW for \$26 and 200 KW for \$18 conditional bids. We assume that we can only buy the whole amounts offered from each user (the bids are not divisible). What would be the outcome of the first-price and the second-price mechanisms? How would we formulate and solve auctions with such generalities?

To be able to address the above example and more complex scenarios with players submitting multiple bids or bid curves, and in the presence of coupling constraints on the amount of electricity to purchase, we consider a more general auction setting.

2.3 VCG mechanism

We consider auctions where each participant can submit multiple bids. A bid in this general setting consists of a price vector c^j and the item (or amount in case of divisible bid) the bidder is buying (or selling in a reverse auction) denoted by m^j. Let N^j denote the number of bids of player j. Hence, $x^j = (c^j, m^j) \in \mathbb{R}^{N^j \times 2}$, where $c^j, m^j \in \mathbb{R}^{N^j}$. Let $M = \sum_{j=1}^{N} N^j$. Let $x \in \mathbb{R}^{2M}$ be denote the decision of all players stacked together as a vector, namely, $x = (c^1, m^1, \ldots, c^N, m^N)$. We also define $c = (c^1, \ldots, c^N)$, $m = (m^1, \ldots, m^N)$. Note that a generalization to bid curves $c^j : \mathbb{R}_+ \to \mathbb{R}_+$ is straightforward. We do not discuss it here for simplicity in notation, but you can find it in [6, 7].

A mechanism is defined by a *choice function* $\delta : x \mapsto \{0,1\}^M$ and a *payment rule* $p : x \mapsto \mathbb{R}^N$. The choice function indicates which bids are accepted, in particular, those that get mapped to 1. The payment rule determines the payment made by (or given to, in a reverse auction) each player. Given $\delta \in \{0,1\}^M$, the total value (or cost in a reverse auction) is $c^T\delta$ (the super-index T is denoting transpose of a vector, whereas the super-indices j, i denote a particular player j, i respectively). There may exist some constraints on the choice function. In Exercise 5 for example, the constraint was to obtain 800 KW of electrical power, $m^T\delta = 800$. Furthermore, the bids from each player were mutually exclusive, $\mathbf{1}^T\delta^j \leq 1$, where $\mathbf{1}$ is a $\mathbb{R}^{N^j}$ dimensional vector of ones, and $\delta^j \in \{0,1\}^{N^j}$ is the choice function restricted to j. We write such constraints in general as $(\delta, m) \in \mathcal{C}$, where $\mathcal{C}$ denotes a constraint set. Note that we consider constraints only on the amount of bids m_j's and not on the prices c_j's. This is important for certain properties we will prove for our mechanism below.

Example 1. Formulate the setup in Exercise 5 as a reverse auction using the notation above.

There are certain properties we may require from a mechanism. For example, the bidders should be incentivized to submit truthful bids. This ensures a *socially efficient* choice, when the operator optimizes the total procurement cost, since the true costs rather than inflated costs are being minimized. The Vickrey-Clarke-Groves mechanism can achieve this goal in the above general auction setting.

The Vickrey-Clarke-Groves auction mechanism is a generalization of the second-price auction to the cases of multiple possibly non-divisible items. It is named after three economists who successively (1961, 1971, 1973) generalized the idea of the second-price auction. To define the VCG mechanism in our current setting, we consider the objective function which chooses the bids that satisfy the constraint while procuring items from bidders who offer the best price $J^\star(x) = \max_{(\delta,m)\in\mathcal{C}} c^T\delta$ (replace max with min in a reverse auction) and the optimizer is denoted by $\delta^\star(x) = \arg\max_{(\delta,m)\in\mathcal{C}} c^T\delta$. Here, the superscript T denotes the transpose. If bidders are being truthful, then this social choice function enjoys the *social welfare maximization* property. Let $J^\star(x^{-j})$ denote the optimal cost when the bids of player j are removed from the optimization problem.

Exercise 6. Show that $J^\star(x^{-j}) \le J^\star(x)$ in a forward auction ($J^\star(x^{-j}) \ge J^\star(x)$ in a reverse auction).

Note that any result below that is defined with respect to a forward auction can be trivially extended to reverse auctions by replacing max with min in the optimization and by appropriately defining the utilities of players with respect to payments received (instead of payments made).

Definition 3. The VCG mechanism choice function and payment rule are defined as

$$\delta^\star(x) = \arg \max_{(\delta,m)\in\mathcal{C}} c^T\delta \tag{1}$$

$$p^j(x) = J^\star(x^{-j}) - (J^\star(x) - (c^j)^T\delta^{\star j}), \quad \forall j \in \mathcal{N}. \tag{2}$$

Exercise 7. Show that for a single non-divisible item forward auction, similar to the one considered in the first example, the VCG mechanism is equivalent to the second-price auction.

Solution. Notice that in a single item auction with each player submitting a single bid, we can consider x^j to be the price the player offers for the item (no need to introduce m^j).
Let $c^j = x^j$ based on the above observation. The VCG choice function (1) is equivalent to

$$\delta^\star(x) = \arg \max_{\delta\in\{0,1\}^N} c^T\delta = \arg\max_j x^j,$$

which is the choice function corresponding to the second-price auction. For the payment rule, consider two cases of player j being the winner or not. In the first case, $\delta^{\star j}(x) = 1$ for player j and $\delta^{\star i}(x) = 0$ for all other players $i \neq j$. Consequently, $J^\star(x) = c^{jT}\delta^{\star j}$. Hence, the payment rule (2) reduces to $p^j(x) = J^\star(x^{-j})$. Note that $J^\star(x^{-j}) = \max_{i\neq j} x^j$. Hence, we have the same payment as the second-price auction. In the case in which j is a losing bidder, $\delta^{\star j}(x) = 0$. Furthermore, $J^\star(x^{-j}) = J^\star(x)$. Consequently, we have that $p^j(x) = 0$, consistent with the second-price auction.

Exercise 8. Determine the VCG winners and the payments in Exercise 5.

Solution. The winners are player 2 for 600 KW and player 3 for 200 KW. To determine the payment made to each winner, we use formula (2):

$$\begin{aligned} p^2(x) &= J^\star(x^{-2}) - (J^\star(x) - (c^2)^T x^{\star 2}) \\ &= 51 - (48 - 30) = 33 \\ p^3(x) &= J^\star(x^{-3}) - (J^\star(x) - (c^3)^T x^{\star 3}) \\ &= 49 - (48 - 18) = 19. \end{aligned}$$

In defining the VCG payments we assume that the two optimization problems in computing the payment (2) are feasible. What does this assumption mean in practice? Is it reasonable?

Let us denote truthful bidding by $(c^j, m^j) \in \mathbb{R}^{N_j\times 2}$ and non-truthful bidding by $(\tilde{c}^j, m^j)$. This means that the player is submitting different prices for the items (in the above, the power she/he can provide). The following result is a generalization of the incentive-compatibility property of the second-price auction shown in Proposition 1.

Proposition 2. *Truthful bidding is a dominant strategy Nash equilibrium in the VCG mechanism.*

Proof. For an arbitrary player j consider the truthful bid $t^j = (c^j, m^j)$ and any other bid $\tilde{x}^j = (\tilde{c}^j, m^j)$. Let $t \in \mathbb{R}^{2M}$ denote the bids where player j places her truthful bid and other players place any arbitrary bid x^{-j}. Furthermore, $\tilde{x}$ denotes the bid profile where player j places $\tilde{x}^j$ and other players place x^{-j}. Let $J^\star(t)$ and $\delta^\star(t)$ denote the optimal value and optimizer corresponding to t. Similarly, define $J^\star(\tilde{x})$ and $\delta^\star(\tilde{x})$ as the optimal value and the optimizer corresponding $\tilde{x}$. Recall the VCG payment rule (2). So, player j's utility under truthful bidding is

$$\begin{aligned} J^j(t) &= c^{jT}\delta^{\star j}(t) - p^j(t) \\ &= c^{jT}\delta^{\star j}(t) - \big(J^\star(x^{-j}) - (J^\star(t) - c^{jT}\delta^{\star j}(t))\big) \\ &= J^\star(t) - J^\star(x^{-j}). \end{aligned}$$

The utility under the bid $\tilde{x}$, on the other hand is

$$\begin{aligned} J^j(\tilde{x}) &= c^{jT}\delta^{\star j}(\tilde{x}) - p^j(\tilde{x}) \\ &= c^{jT}\delta^{\star j}(\tilde{x}) - \big(J^\star(x^{-j}) - (J^\star(\tilde{x}) - \tilde{c}^{jT}\delta^{\star j}(\tilde{x}))\big) \\ &= c^{jT}\delta^{\star j}(\tilde{x}) - \big(J^\star(x^{-j}) - \sum_{i\neq j} c^{iT}\delta^{\star i}(\tilde{x})\big) \\ &= c^{jT}\delta^{\star j}(\tilde{x}) + \sum_{i\neq j} c^{iT}\delta^{\star i}(\tilde{x}) - J^\star(x^{-j}). \end{aligned}$$

Now, notice that $c^{jT}\delta^{\star j}(\tilde{x}) + \sum_{i\neq j} c^{iT}\delta^{\star i}(\tilde{x}) \leq J^\star(t)$ because $\delta^\star(\tilde{x})$ is a feasible suboptimal allocation for the bid profile t (it is optimal for the bid profile $\tilde{x}$ and not necessarily for bid profile t). Hence, we have $J^j(\tilde{x}) \leq J^j(t)$ as desired. Since the proof was for an arbitrary player j and for any x^{-j}, we conclude that truthful bidding is the weakly dominant strategy Nash equilibrium. □

Observe that if the bid prices $\tilde{c}$ were used in the definition of the constraints, then the argument that $\delta^*(\tilde{x})$ is a feasible allocation for the problem $\max_{(\delta,m)\in\mathcal{C}} c^T\delta$ would not readily hold.

Exercise 9. Verify that the above proposition is true if $J^\star(x^{-j})$ in (2) is replaced by any other positive function $h(x^{-j})$. The main insight in definition of the VCG payment rule is that the payment should not depend on player j's bid to ensure incentive compatibility. The choice of $h(x^{-j}) = J^\star(x^{-j})$ is referred to as the Clarke Pivot rule. This choice ensures an additional desirable property of the mechanism, which an arbitrary h may not do. In particular, using $J^\star(x^{-j})$ ensures the minimum total payment to each player (in a reverse auction) resulting in non-negative utility (in markets it is known as price recovery). In game theory, the property of having a non-negative utility is referred to as individual rationality because it ensures it is rational for a player to participate in the auction (see Proposition 3).

Proposition 3. *At the dominant strategy incentive compatible Nash equilibrium of the VCG mechanism, each player obtains a non-negative utility.*

Proof. Note that $J^\star(x^{-j}) \leq J^\star(x)$ because the former maximization has one fewer player. The utility under truthful bidding is $J^j(x) = J^\star(x) - J^\star(x^{-j})$. It follows that $J^j(x) \geq 0$. □

The VCG mechanism also ensures the utility of all participants including the auctioneer is maximized. This property is referred to as social welfare optimization and is shown here. Note that the utility of the auctioneer at a give bid x of all players is $J^0(x) = \sum_{j=1}^N \big[J^\star(x^{-j}) - (J^\star(x) - c^{jT}\delta^{\star j}(x))\big]$.

Proposition 4. *At the dominant strategy incentive compatible Nash equilibrium of the VCG market, the sum of utilities of all players (including the auctioneer) is maximized.*

Proof. The utility of player j under truthful bidding is $J^j(t) = c^{jT}\delta^{\star j}(t) - \big(J^\star(t^{-j}) - (J^\star(t) - c^{jT}\delta^{\star j}(t))\big)$. The utility of the auctioneer under all submitting truthful bid is $J^0(t) = \sum_{j=1}^N \big[J^\star(t^{-j}) - (J^\star(t) - c^{jT}\delta^{\star j}(t))\big]$. Adding these utilities we obtain $J^0(t) + \sum_{j=1}^N J^j(t) = J(t) = \max_{(\delta,m)\in\mathcal{C}} c^T\delta$. □

Unfortunately, the VCG mechanism suffers from several drawbacks.

Example 2. Consider purchase of 800 KW of electrical power as before. There are five providers. Player 1 offers 800 KW at $50. The next four players each offer 200 KW for $0. What is the VCG outcome? What is the total payment made by the auctioneer?

The above highlights some of the pathologies of the VCG mechanism. In particular, note that the auctioneer ends up paying a very high price despite the fact that it is possible to procure electricity at a much lower price. Furthermore, the example shows shill bidding and collusion can occur in a VCG mechanism and that the auctioneer's total payment is non-monotone as a function of the number of bidders. Shill bidding means a bidder can submit bids with multiple identities. Collusion means bidders can join forces to increase their utilities. Finally, payoff monotonicity means that as the number of players increases (more competition in the market), the auctioneer's utility should not decrease. This is not the case for VCG. For a detailed example of these pathologies please see [2]. So, it seems we are facing some trade-offs. On the one hand, the VCG ensures the dominant strategy Nash equilibrium is bidding truthfully. Namely, players in the VCG mechanism have no incentive for *unilateral* deviation from a truthful strategy. However, they can have incentives to collectively deviate from the truthful mechanism. Are there cases in which the VCG mechanism may not suffer from the above pathologies? We will discuss this briefly in the next subsection.

2.4 A brief introduction to core

Some of the problems of the VCG mechanism can be avoided by ensuring that the VCG utilities are in a set referred to as the *core*. The core is a concept from coalitional game theory. In the following, we will bring its definition.

For every $\mathcal{S} \subseteq \mathcal{N}$, let $J^\star(\mathcal{S};t)$ be the objective function under participation of players in the coalition $\mathcal{S}$. It is defined by the following expression:

$$J^\star(\mathcal{S};t) = \max_{\substack{(\delta,m)\in\mathcal{C},\\ \delta^j=0,\forall j\notin S}} c^T\delta.$$

Recall that a function $f : 2^{\mathcal{N}} \to \mathbb{R}_+$ is monotone if for every $S_1, S_2 \in 2^{\mathcal{N}}$, $S_1 \subset S_2 \implies f(S_1) \le f(S_2)$. We can conclude the monotonicity of the set function $J^\star(\cdot;t) : 2^{\mathcal{N}} \to \mathbb{R}_+$ using a similar reasoning to that of Exercise 6. Next, we bring in the definition of the core.

Definition 4. The core, $\mathcal{K}(t)$, is defined by

$$\mathcal{K}(t) = \Big\{[J^0, J^1, J^2, \ldots, J^N] \in \mathbb{R}_+^{N+1} \mid J^0 + \sum_{j\in\mathcal{N}} J^j = J^\star,$$
$$J^0 + \sum_{j\in\mathcal{S}} J^j \ge J^\star(\mathcal{S};t),\ \forall \mathcal{S} \subset \mathcal{N}\Big\},$$

where above, $J^\star = \max_{(\delta,m)\in\mathcal{C}} c^T\delta$.

The non-negativity constraints on the utilities ensure individual rationality. The equality constraint corresponds to social welfare maximization, since the sum of utilities are maximized at $J^\star$. Finally, the inequality constraints translate into the following: there are no coalition of bidders $\mathcal{S} \subset \mathcal{N}$ that can make a special deal with the auctioneer and end up with a better total utility.

The VCG mechanism is called core-selecting if its utility profiles lie in the core under any possible truthful bids and under any set of participants, that is,

$$[J^0, J^1, \dots, J^N] = \left[\sum_{j=1}^{N} \left(J^\star(t^{-j}) - (J^\star(t) - c^{jT}\delta^{\star j}(t))\right), J^\star(t) - J^\star(x^{-1}), \dots, J^\star(t) - J^\star(x^{-N})\right] \in \mathcal{K}(t),$$

for all $t \in \mathbb{R}_+^{2M}$. For the sake of simplicity in notation, the mathematical condition ignores the requirement "*under any set of participants*".

Proposition 5 ([6],Theorem 3–*(ii)*)**.** *If the VCG mechanism is core-selecting, then shill bidding is not profitable.*

Unfortunately, one needs to place additional assumptions on the bids or constraint function $\mathcal{C}$ in a general auction to ensure the outcome of the VCG mechanism are in the core [6]. You can verify that in Example 2, the inequality constraints of core are violated for $\mathcal{S} = \{1\}$. If we modify that example such that the mutually exclusive bids are equally spaced and marginally nondecreasing, the VCG mechanism becomes core-selecting.

Example 3. Consider purchase of 800 KW of electrical power as before. There are five providers. Player 1 offers 800 KW at \$50, but also 600 KW at \$37.5, 400 KW at \$25, and 200 KW at \$12.5. The next four players each offer 200 KW for \$0. What is the VCG outcome? What is the total payment made by the auctioneer? Can you verify that the utility profile lies in the core?

Alternatively, in a general market setting, one can define payment rules that are immune to collusion and shill bidding [7] by directly choosing their utility profile from the core. However, these mechanisms may not ensure incentive compatibility. Furthermore, since the theoretical analysis becomes very challenging for such general mechanisms, one has to resort to heuristic approaches and simulations to understand the properties (for instance, revenues and incentives) of equilibria.

The topic of mechanism design is relevant not only for auctions, but also for voting systems, item exchanges (read for example regarding student-public school matching in Boston, Kidney exchange auctions, stable marriage problems) and social preference matching. For further details you can consult several of mentioned books on game theory.

2.5 Summary and further readings

We used game theory to formulate auctions and mechanism design problems. For beautiful and detailed exposition of auctions and mechanisms I recommend Chapters 14, 15, 16 of [12] and Chapters 10, 11 of [5]. Though some of the materials there will be clear after we discuss Bayesian games in the next lecture.

2.6 Acknowledgements

I wrote these notes in collaboration with my Ph.D. student Orcun Karaca. Orcun's thesis focused on advancing auction theory and mechanism design for electricity markets. This topic is highly relevant as the electricity markets are undergoing great transition due to the renewable sources of energy integration in the power grid. If you are interested in research on this topic, I recommend Orcun's thesis as a starting point.

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